UNIT - 1

- Introduction to Statically determinate Structure
- Static & kinematic Indeterminacies
- Castigliano's Theorem
- Strain Energy Method.
- Analysis of frames with one or two redundant members using Castigliano's 2nd theorem.

1) Introduction to Statically Indeterminate Structures:

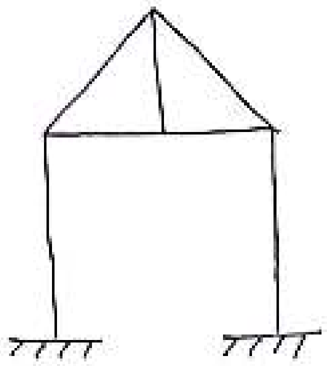
- **Structure :**
  - The primary function of which is to recieve all the external loads of some point and transmit the load safely to some other point called structure.
  - There are three types of structure:
    a) Skeltal Structure
    b) Surface Structure
    c) Solid Structure

a) **Skeltal Structure :**
- These are those structure which can be idealised to a series of straight or curved lines and looks like a skeltal is called skeltal structure.

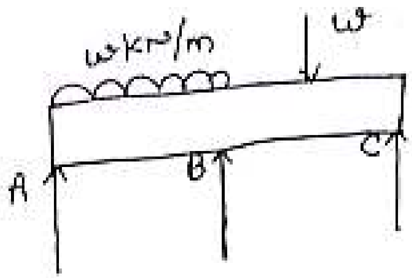
b) **Solid Structure :**
- These are those which can neither be idealised to skeltal nor a surface structure is called solid structure.

c) **Surface Structure :**
- Any structure which can idealised to a plane or curve surface is known as surface structure.

- **Types of Skeltal Structure :**
  i) Pin Jointed
  ii) Rigid Jointed → a) Plane frame structure
  b) Space frame structure



[Skeltal]



[Surface Structure]

- **Statically Determinate Structure :**
  - The structure which can be analysed by using equation of statical equillibrium is called statically determinate structure.
  - For eg : Beams or trusses with both ends simply supported, one end hinged & another on roller & the cantiliver etc.

- **Statically Indeterminate Structure :**
  - A structure which cannot be analysed by using equations of equillibrium only is called statically indeterminate structures.
  - For eg : Fixed beam, Continuous beam, propped cantiliver.
  - It is also known as redundant structure.

- **Equillibrium Conditions of Static Equillibrium :**

a) For 2D Structure [3]

$\Sigma M = 0$

$\Sigma H = 0$

$\Sigma V = 0$

For 3D structures [6]

$\Sigma M_x = 0$ $\Sigma F_x = 0$

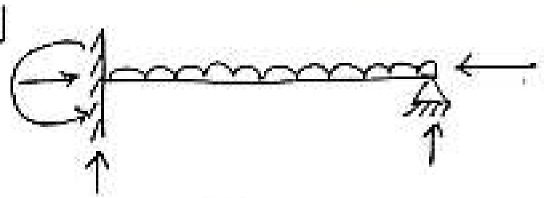
$\Sigma M_y = 0$ $\Sigma F_y = 0$

$\Sigma M_z = 0$ $\Sigma F_z = 0$

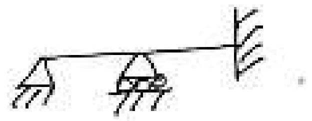
2) Degree of Indeterminacy :

- It can be defined as the no. of additional equation required for determining the unknown reaction of statically indeterminate structure.

- DOI = No of unknown Reactions − Static equillibrium
  = uknown − known conditions.

Eg a) 

DOI = 5 − 3
= 2

b) 

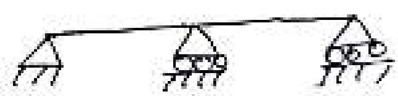
DOI = 6 − 3
= 3

- **Determination of degree of indeterminacy for Beam :**
  - for roller support = 1
  - for Hinged support = 2
  - for fixed support = 3

- for beam structure, there are only three independent equation.

Eg a) 

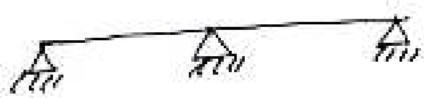
DOI = 6 − 3 = 3

b) 

DOI = 4 − 3 = 1

c) 

DOI = 7 − 3 = 4

d) 

DOI = 6 − 3 = 3

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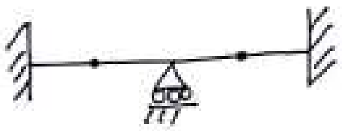
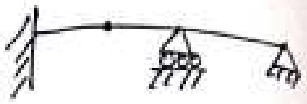
Note: If there is external hinges are provided there will be an additional equillibrium equation [moment at hinge = 0] for each hinge

Eg: a)



$DOI = 7-3-2 = 2$

b)



$DOI = 6-3-1 = 2$

a). Static Indeterminacy [Ds]:

- If equilibrium equations are sufficient to analyse a structure completely for unknown force, it is called statically indeterminate structure.

- $Ds = Dse + Dsi$ – no of force releases.

  Ds = Total Static indeterminacy

  Dse = external indeterminacy [related to support reaction]

  Dsi = internal indeterminacy [related to type of joints & frame]

- **Formulaes for Ds**:

a) For Frame $\boxed{2D}$ → $3m + r - 3J$

OR

$3C - R$

$\boxed{3D}$ → $6m + r - 6J$

m = members

r = support reactions

J = No of Joints.

• for Hinge in member : $D_s - (m'-1)$

$$D_s = 3m + r - 3J - (m'-1)$$

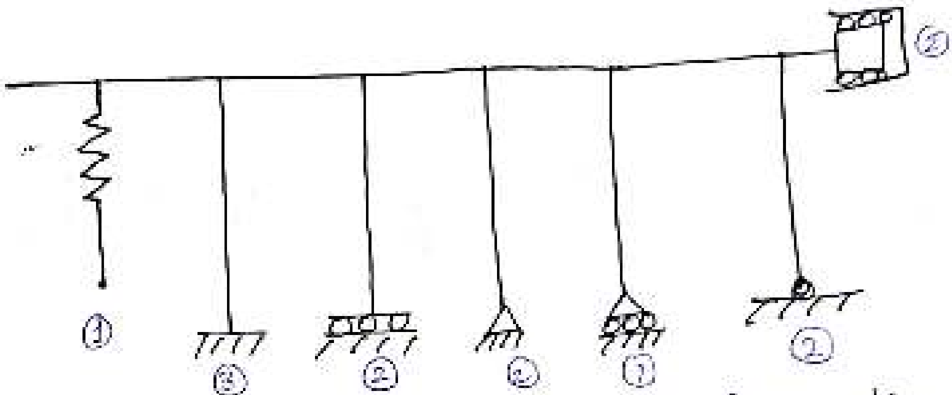
m' = no of member connected to hinge

b) for Pin jointed Truss :

$$D_s = m + r - 2J$$

Examples of Static Indeterminacy :

a)

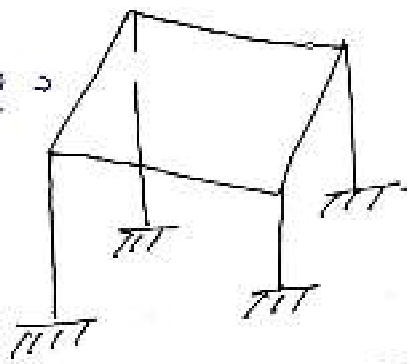


Dse = Support reaction − no of equation

13 − 3 = 10

b)

3D :



$$D_{se} = [4 \times 6] - 6 = 18$$

$$\boxed{\begin{aligned} D_{si} &= 3C \rightarrow \text{Plane frames} \\ & 6C \rightarrow \text{Space frames} \end{aligned}}$$

C → cuts required to convert a closed structure to open structure.

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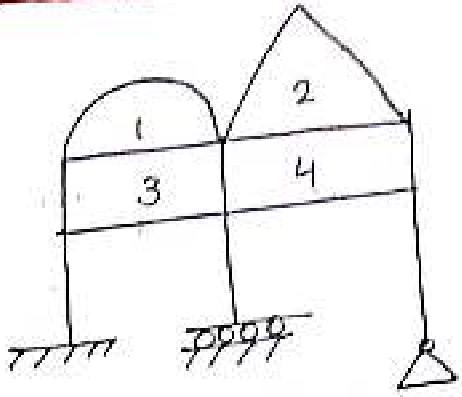
Q.

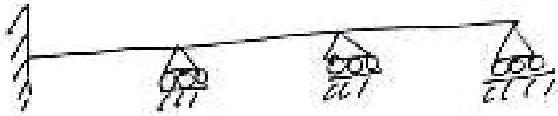


Dse = support reaction - equillibrium eqn
$[3+2+2] - 3 = 4$

Dsi = $3C = 3\times4 = 12$

Ds = Dse + Dsi = $4+12 = 16$

Q



Ds = $3m + r - 3J$
$= [3\times3] + 6 - [3\times4] = 3$

Dse = $6 - 3 = 3$

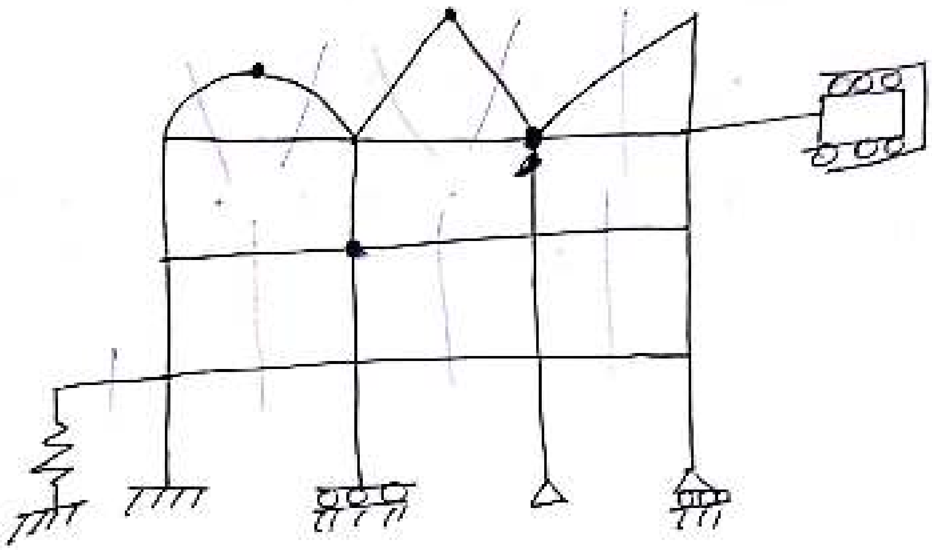
Dsi = Ds - Dse = $\underline{3 - 3 = 0}$

Internally stable

Q

Ds = $3m + r - 3J$
$[3\times1] + 5 - 3(2) = \underline{\underline{2}}$

Q→



$Dse = \text{reaction} - \text{equillibrium eq}^n$

$11 - 3 = 8$

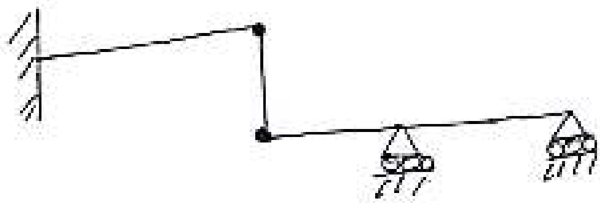
$Dsi = 3C = 3\times9 = 27$

$Ds = [8+27] - (m'-1)$

$= [8+27] - [2-1] - [2-1] - [5-1] - [4-1]$

$= 27 + 8 - 9 = \underline{\underline{26}}$

Q→



$Dse = 5 - 3 = 2$

$Dsi = 3C = 3(0) = 0$

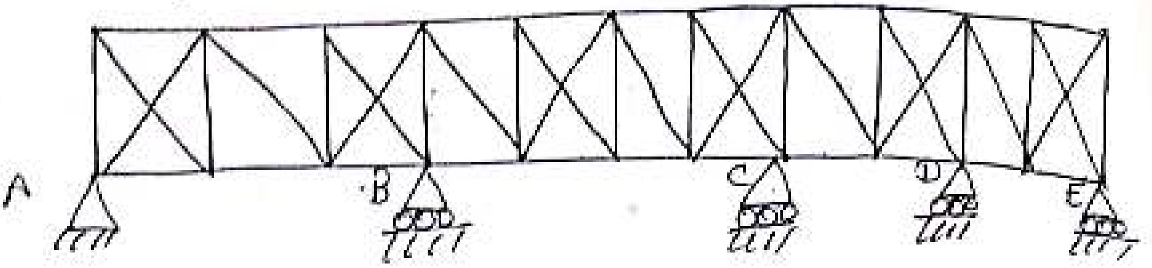
$Ds = Dse + Dsi - (m'-1)$

$= 2 + 0 - (2-1) - (2-1)$

$= 0$

Or $Ds = 3m + r - 3J - (m'-1) = 0$

Q → Analyse Pin Jointed Frame :



$D_{se} = r - 3$

$= 6 - 3 = 3$

$D_{si} = m - (2J - 3)$

$= 51 - (2 \times 24 - 3) = \cancel{54} = 48.$

$51 - 48 + 3 = 6$

$D_s = D_{se} + D_{si}$

$= 6 + 3 = 9$

or

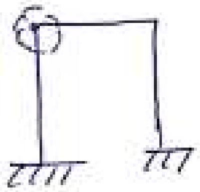
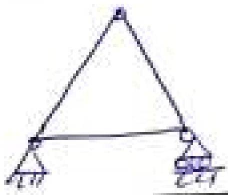
$D_s = m + r - 2J$

$= 51 + 6 - 48 = \underline{9}$

b) Kinematic Indeterminacy : [D_K]

- Also called as degree of freedom.
- The no of unknown joint displacements are called degree of freedom.

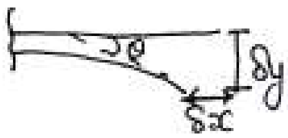
- **Degree of freedom**

Types of Joints	D_K
Rigid joint of a plane frame. 	$3 \, [\delta x, \delta y, \theta]$
Rigid joint of space frame	$6 \begin{bmatrix} \theta x, \theta y, \theta z \\ \delta x, \delta y, \delta z \end{bmatrix}$
~~Rigid~~ Pin Joint of plane frame 	$2 \, [\delta x, \delta y]$
Pin Joint of a space frame	$3 \, [\delta x, \delta y, \theta_z]$

Types of Support

(D_k) Degree of freed

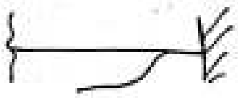
- Free end — $3\ [\delta x, \delta y, \theta]$



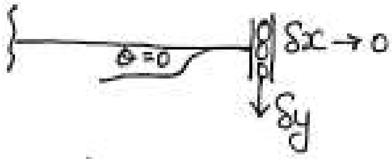
- Roller Support — $2\ [\theta, \delta x]$

- Hinged / Pinned support — $1\ [\theta]$

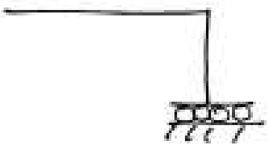
- Fixed Support — ~~[δy]~~ 0



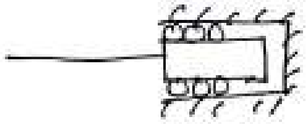
- Vertical Shear Hinge — $1\ [\delta y]$



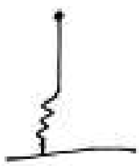
- Horizontal shear hinge — $1\ [\delta x]$



- Damper Support — $1\ [\delta x]$

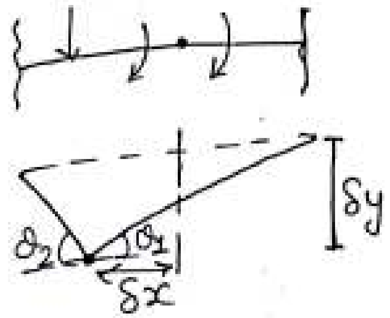


- Spring Support — $2\ [\theta, \delta x]$



Effect of force releases on DOF [Dk]:

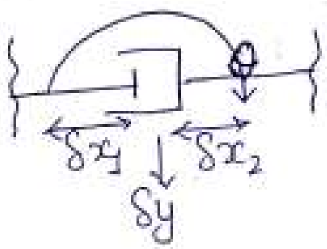
1. Internal moment hinge



$4\left[\begin{array}{l} 2 \text{ rotations } \theta_1 \& \theta_2 \\ 2 \text{ translations } \delta x \& \delta y \end{array}\right]$

<u>Note</u>: Each member connected to a hinge can have its own rotation, in addition to δx & δy.

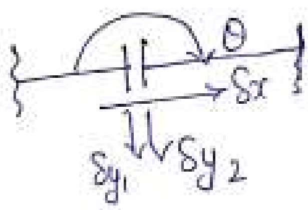
2. <u>Horizontal Shear Release</u>:



4 [DOF]

$\left[\begin{array}{ll} 2 & \text{horizontal } \delta x_1, \delta x_2 \\ 1 & \delta y \end{array}\right]$

3. <u>Vertical Shear release</u>:



$4\left[\begin{array}{l} \text{DOF} \\ \delta y_1, \delta y_2, \delta x \& \theta \end{array}\right]$

<u>Formulaes</u>

- <u>Frame</u>: $[D_K = 3J - R] \rightarrow$

  $D_K = 3J - R - m$ if axially Inflexible/Rigid

  - In hinge within member $D_K + 4$
  - If Joint hinge $D_K + (m' - 1)$

- <u>Truss</u> = $D_K = 2J - R$ (2D)

  $D_K = 3J - R$ (3D)

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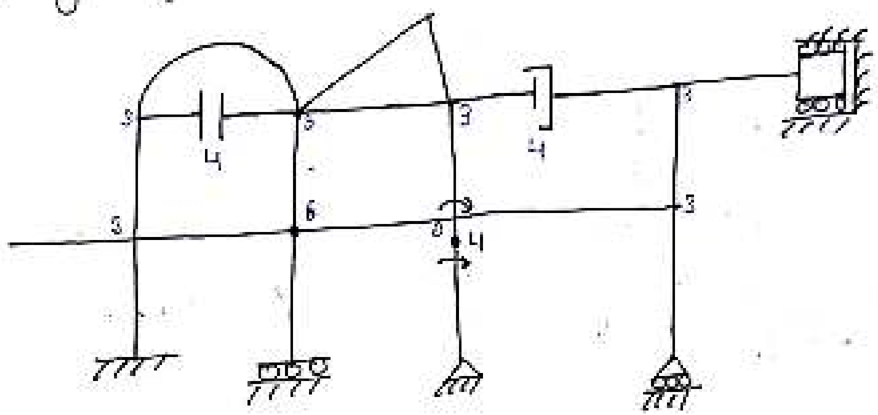
• D_K of rigid jointed Plane frame :



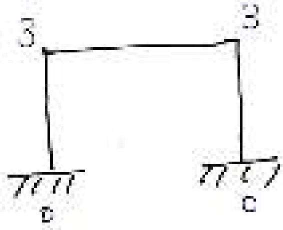
• ~~D_K = ... 52 (con ... 2J + 4 + (m' − 1)~~

$D_K = 52$ [considering axial deformations]

$D_K = 52 -$ total no of members

$= 52 - 22 = 30$ [neglecting axial deformation]

Q. Find D_K when only beam is rigid

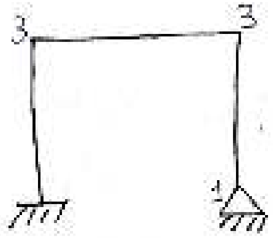


$D_K = 6 - 1 = 5$ [if beam]

$D_K = 6 - 2$ [If column rigid]

$D_K = 6 - 3$ [If all members are rigid]

Q.



$D_K = 7$ [considering axial deformation]

$D_K = 7 - 3 = 4$

[neglecting ~~action~~ axial deformation].

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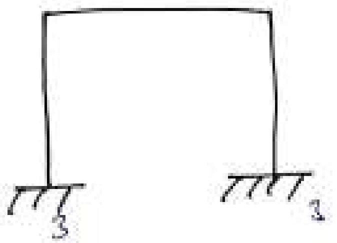
DK for Pin Jointed Plane frame :

$$DK = NJ - C$$

N = 3 → Rigid Jointed Plane frame   N = DOF at a joint
N = 6 → " " Space frame   J = No of Joint
N = 2 → Pin Jointed Plane frame   C = Compatability eqⁿ
N = 3 → " " Space frame.

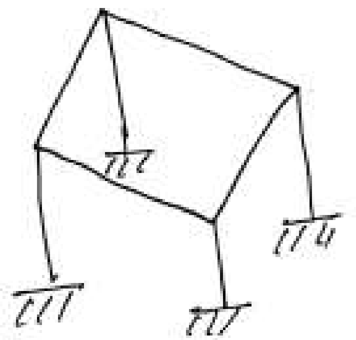
Q.



J = 4
N = 3
C = 6

DK = 12 − 6 = 6

Q.



J = 4
N = 6
DK = 24 [considering axial deformⁿ]

DK = 24 − 8
= 16 [neglecting axial deformation]

3] Strain Energy Method :

a). **Strain Energy :**

- Energy stored due to straining of a body.

b). **Resillience :**

- Energy stored in the material within the elastic limit.

c). **Proof Resillience :**

- Energy stored in the material upto elastic limit.
- It is max strain energy.

d). **Modulus of Resillience :**

- It is proof resillience per unit volume.

e). **Toughness :**

- Energy stored upto fracture point.

f) **Modulus of Toughness :**

- Toughness per unit volume.

- **Strain Energy Tension or compression :**

(a] Neglecting the weight of bar :

- Consider a small element of a bar, length [ds]. If a graph drawn of load against elastic extension the shaded area under graph gives work done & hence the strain energy.

$$U = \frac{1}{2} P \delta \rightarrow ①$$

But young's modulus = ~~E = Pds~~ $E = \frac{Pds}{A\delta}$

$$\delta = \frac{Pds}{AE} \rightarrow ②$$

Substitute eqⁿ ② in ①

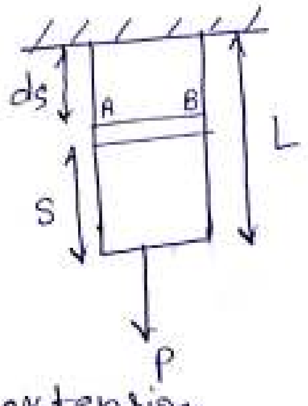
For bar element = $U = \frac{P^2 ds}{2AE}$

∴ Total strain energy for a bar of length L, = $U = \int_0^L \left(\frac{P^2 ds}{2AE}\right)$

$$\boxed{U = \frac{P^2 L}{2AE}}$$

b) **Including weight of bar** :-

- Consider now a bar of length L mounted vertically. At any section AB the total load on the section will be external load P together with weight of bar material below AB.



Total load AB = $[P \pm \rho g A s]$

$\left[\delta = \frac{Pds}{AE}\right]$ $\delta = \frac{[P \pm \rho g A s] ds}{AE}$

But work done = $\frac{1}{2} \times$ Load $\times$ extension

$$= \frac{1}{2} \times (P \pm \rho g A s) \times \frac{[P \pm \rho g A s] ds}{AE}$$

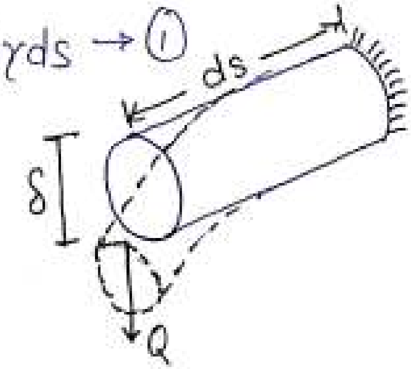
$$= \frac{P^2}{2AE}ds + \frac{P\rho g}{E}s\,ds + \frac{(\rho g)^2 A}{2E}s^2 ds$$

Total strain energy or work done is

$$\int_0^L \frac{P^2}{2AE}ds + \int_0^L \frac{P\rho g s\,ds}{E} + \int_0^L \frac{(\rho g)^2 A}{2E}s^2 ds$$

$$\boxed{U = \frac{P^2 L}{2AE} + \frac{P\rho g L^2}{2E} + \frac{(\rho g)^2 A L^3}{6E}}$$

- **Strain Energy due to shear :-**
  - Consider the element bar now subjected to a shear load Q at one end causing deformation through the angle γ (the shear strain) & a shear deflection.



Strain energy $U = \frac{1}{2}Q\delta = \frac{1}{2}Q\gamma ds \rightarrow (1)$

But modulus of rigidity

$$G = \frac{\tau}{\gamma} = \frac{Q}{\gamma A}$$

$$\gamma = \frac{Q}{AG} \longrightarrow (2)$$

Substitute (2) in (1)

$$U = \frac{1}{2}Q\frac{Q}{AG}ds = \frac{Q^2 ds}{2AG}$$

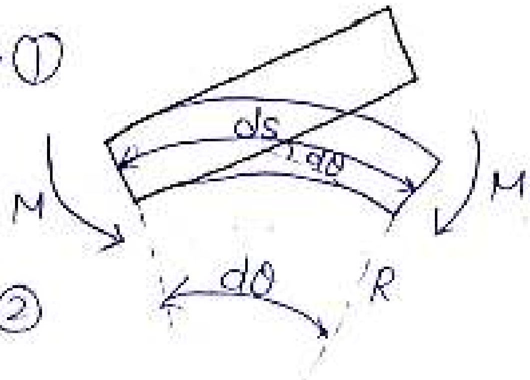
Total Strain energy $= U = \int_0^L \frac{Q^2 ds}{2AG}$

$$\boxed{U = \frac{Q^2 L}{2AG}}$$

Strain Energy - Bending :

- Let the element now be subjected to a constant BM - M causing it to bend into an arc of radius R & subtending an angle $d\theta$ at the centre. The beam will also have moved through an angle $d\theta$.

Strain energy $= \frac{1}{2} M \times d\theta \rightarrow (1)$



But $ds = R\,d\theta$ & $\frac{M}{I} = \frac{E}{R}$

$$d\theta = \frac{ds}{R} = \frac{M\,ds}{EI} \rightarrow (2)$$

Substitute (2) in (1)

$$\text{Strain energy} = \frac{1}{2} M \frac{M\,ds}{EI} = \frac{M^2 ds}{EI}$$

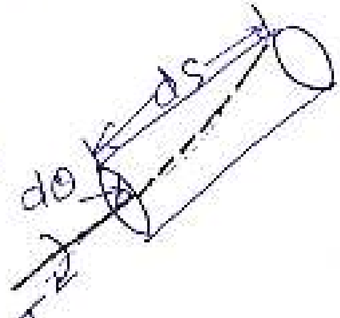
$$\text{Total strain energy} = \quad U = \int_0^L \frac{M^2 ds}{EI}$$

$$\boxed{U = \frac{M^2 L}{2EI}}$$

Strain energy - Torsion :

- The element is now considered subjected to Torque T producing angle of twist $d\theta$ radians.

Strain energy $= \frac{1}{2} T\,d\theta \rightarrow (1)$



But from simple torsion theory

$$\frac{T}{J} = \frac{G d\theta}{ds} \quad \& \quad d\theta = \frac{T\,ds}{GJ} \rightarrow (2)$$

Substitute (2) in (1)

Total strain energy $U = \int_0^L \frac{T^2 ds}{2GJ}$

$$\boxed{U = \frac{T^2 L}{2GJ}}$$

- **Castigliano's first theorem** :-

- The concept of elastic strain energy can be very useful in the study of deflection of various points of structure under load.

- Instead of directly equating the external work to the internal strain energy, considerable simplification is obtained by castigliano's first theorem which states that the deflection caused by any external force is equal to the partial derivative of the strain energy w.r.t that force.

- If there is any elastic system in equilibrium under the action of a set of forces $W_1, W_2, W_3 \ldots W_n$ and corresponding displacements $\delta_1, \delta_2 \ldots \delta_n$ and set of moments $M_1, M_2, M_3 \ldots M_n$ and corresponding rotations $\phi_1, \phi_2 \ldots \phi_n$ the partial derivative of total strain energy U wrt any one of the forces or moment taken individually would yield its corresponding

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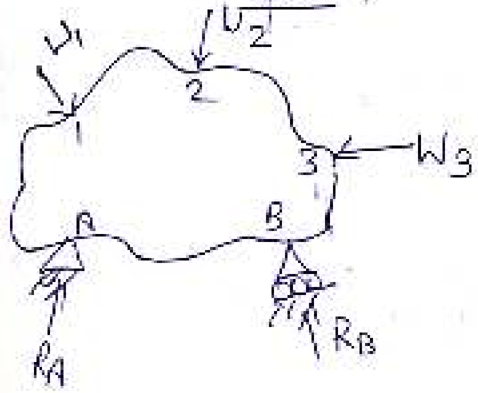
displacement in its direction of action

$$\frac{\delta U}{\delta W_1} = \delta_1 \qquad \frac{\partial U}{\partial W_1} = \delta_1$$

$$\frac{\delta U}{\delta W_2} \qquad \frac{\partial U}{\partial M_1} = \phi_1$$



• **Deflection of Beams by castigliano's 1st theorem:**

a) Axial force

$$U = \int_0^L \frac{P^2 dx}{2AE}$$

$$\delta_1 = \frac{\partial U}{\partial W_1} = \int_0^L \frac{P}{AE} \cdot \frac{\partial P}{\partial W_1} \cdot dx$$

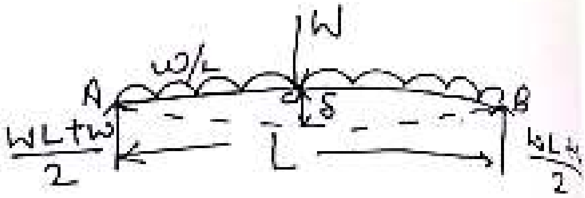
b) Strain Energy Due to Bending:

$$U = \int_0^L \frac{M^2 dx}{2EI}$$

Deflection $$\delta_1 = \frac{\partial U}{\partial W_1} = \int_0^L M \left(\frac{\partial M}{\partial W_1}\right) \frac{dx}{EI}$$

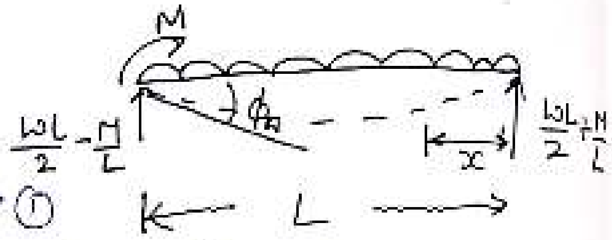
Rotation $$\phi_1 = \frac{\partial U}{\partial M_1} = \int_0^L M \left(\frac{\partial M}{\partial M_1}\right) \frac{dx}{EI}$$

Q-1 Calculate the central deflection and slope of a simply supported beam carrying UDL w/l over whole span.



Sol Central Deflection

$$\delta_C = \frac{\partial U}{\partial W}$$



$$= \frac{1}{EI}\int_0^L M_x \frac{\partial M_x}{\partial W}\,dx \quad \rightarrow (1)$$

M_x = bending moment at x from A

$$M_x = -\left(\frac{wL}{2} + \frac{W}{2}\right)x + \frac{wx^2}{2}$$

$$\frac{\partial M_x}{\partial W} = -\frac{x}{2}$$

Substitute in (1)

$$\delta_C = \frac{2}{EI}\int_0^{L/2}\left[\left(\frac{wL}{2}+\frac{W}{2}\right)x - \frac{wx^2}{2}\right]\frac{x}{2}\,dx$$

Put $W = 0$

$$\delta_x = \frac{2}{EI}\int_0^{L/2}\left[\frac{wLx}{2} - \frac{wx^2}{2}\right]\frac{x}{2}\,dx$$

$$= \frac{2}{EI}\left[\frac{wLx^2}{12} - \frac{wx^4}{16}\right]_0^{L/2} = \frac{5}{384}\frac{wL^4}{EI}$$

Slope at ends

$$\phi_A = \frac{\partial U}{\partial M} = \frac{1}{EI}\int_0^L M_x \cdot \frac{\partial M_x}{\partial M}\,dx \quad \rightarrow (2)$$

$$M_x = -\left[\frac{wL}{2} + \frac{M}{L}\right]x + \frac{wx^2}{2}$$

$$\frac{dM_x}{dM} = -\frac{x}{L}$$

Substitute in (2)

$$\phi_A = \frac{1}{EI}\int_0^L \left[\int\left(\frac{WL}{2} + \frac{M}{2}\right)x - \frac{wx^2}{2}\right]\frac{x}{L}\,dx$$

Put $M = 0$

$$\phi_A = \frac{1}{EI}\int_0^L \left(\frac{wL}{2}x - \frac{wx^2}{2}\right)\frac{x}{L}\,dx$$

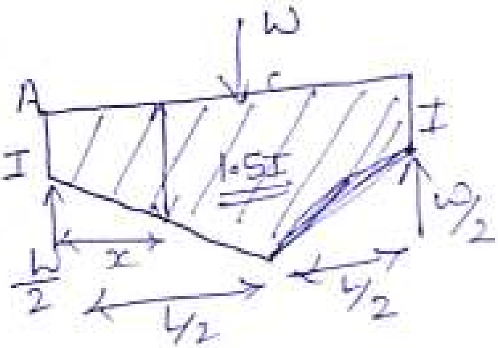
$$= \frac{1}{EI}\left[\frac{wx^2}{6} - \frac{wx^4}{8L}\right]_0^L = \frac{wL^3}{24EI}$$

Q2 A freely supported beam of span L carries a central load w. The sectional area of beam is so designed that the moment of Inertia of section increases uniformly from I at ends to 1.5I at mid. Calculate central deflection.

Sol

Deflection is given by

$$\delta_C = \frac{\partial U}{\partial W} = \frac{1}{E}\int \frac{M_x}{I_x}\cdot\frac{\partial M_x}{\partial W}\,dx$$



In the above integral

$$I_x = I\left(1 + \frac{x}{L}\right) \text{ and is function of } x$$

$$M_x = -\frac{W}{2}x \text{ from } x = 0 \text{ to } x = L/2$$

$$\frac{\partial M_x}{\partial W} = -\frac{x}{2}$$

Substitute value in eqn δ_C

$$\delta_C = \frac{2}{E}\int_0^{L/2} \frac{1}{I\left(1 + \frac{x}{L}\right)}\frac{W}{2}x\cdot\frac{x}{2}\,dx$$

$$= \frac{W}{2EI}\int_0^{L/2} \frac{x}{\left(1 + \frac{x}{L}\right)}\,dx$$

Substitute $x + L = t$

$$\delta x = \frac{WL}{2EI} \int_{L}^{3L/2} \left(t - 2L + \frac{L^2}{t} \right) dt$$

$$= \frac{WL}{2EI} \left[\frac{t^2}{2} - 2Lt + L^2 \log_e t \right]_{L}^{3L/2}$$

$$= 0.015 \frac{WL^3}{EI}$$

- <u>Minimum Strain Energy</u>:

- Castigliano states that among all the statically possible states of stress in a structure subjected to a variation of stress during which the condition of equilibrium are maintained, the correct one is that which makes the strain energy of system a minimum.

- $$\frac{\partial U}{\partial R_1} = 0 \qquad U = \text{Strain energy}$$

$$\frac{\partial U}{\partial R_2} = 0 \qquad R_1 \,\&\, R_2 \text{ redundant forces.}$$

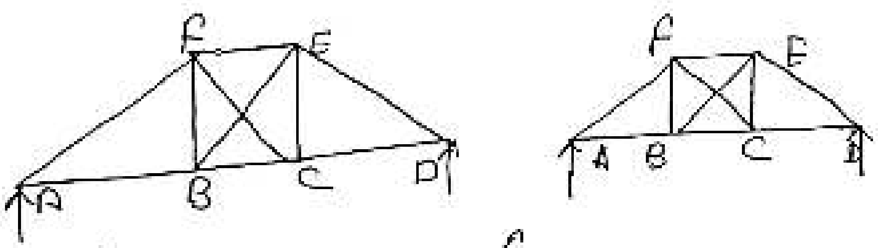
Castigliano's 2nd Theorem :

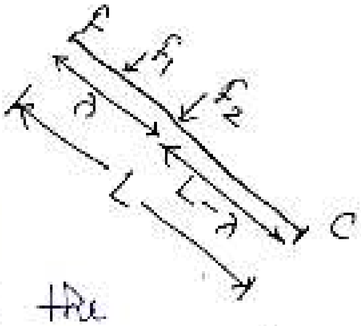
- Let us consider expression for strain energy as a function of the external loads/forces & assume that the principal of superposition holds, in which case the strain energy is a homogenous function of 2nd degree.

- "Under such condition, a partial derivative of strain energy w.r.t one of external forces gives the displacement corresponding to that force." This statement is usually called Castigliano's 2nd theorem.

$$\boxed{\frac{\partial U}{\partial T} = \lambda}$$

- Proof :





- Consider a redundant frame in which FC is a redundant member of geometrical length L. Let the actual length of member FC be $(L-\lambda)$. λ being the initial lack of fit.

- According to hooke's law

$$f_2 f_1 = \text{deformation} = \frac{T(L-\lambda)}{AE} = \frac{TL}{AE} \text{ approx}$$

$T =$ tensile force

$$ff_1 = ff_2 - f_1 f_2 = \lambda - \frac{TL}{AE} \longrightarrow ①$$

- Let the member FC be removed & consider tensile force T acting at corners F & C. So that basic system is not changed

Now ff_1 = relative deflection of F & C

$$= \frac{\partial U'}{\partial T} \longrightarrow ②$$

$= U' =$ strain energy of whole frame except that of member FC.

Equating ① & ②

$$\frac{\partial U'}{\partial T} = \lambda - \frac{TL}{AE}$$

The $\quad \frac{\partial U'}{\partial T} + \frac{TL}{AE} = \lambda \longrightarrow ③$

- Strain energy stored in member FC due to force T is

$$U_{FC} = \frac{1}{2} T \cdot \frac{TL}{AE} = \frac{T^2 L}{2AE}$$

$$\frac{\partial U_{FC}}{\partial T} = \frac{TL}{AE}$$

Substitute $\frac{TL}{AE}$ in eq^n ③

$$\frac{\partial U'}{\partial T} + \frac{\partial U_{FC}}{\partial T} = \lambda$$

$$\boxed{\frac{\partial U}{\partial T} = \lambda}$$

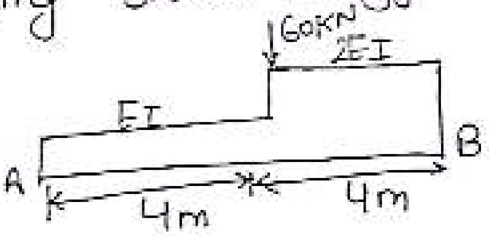
if there is no initial lack $\lambda = 0$

$$\boxed{\frac{\partial U}{\partial T} = 0}$$

Numericals

→ Castigliano's 1st Theorem :

Q. Determine the deflection under 60kN load in beam using strain energy method.



Sol. • $R_A + R_B = 60\,kN$

$R_A = R_B = 30\,kN$

• Considering a distance of (x) from (A)

$M_x = 30x$

$$U = \int_0^4 \frac{M^2}{EI}dx + \int_0^4 \frac{M^2}{2EI}dx$$

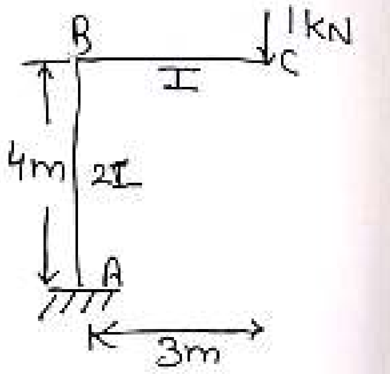
$$= \int_0^4 \frac{(30x)^2}{2EI}dx + \int_0^4 \frac{(30x)^2}{2EI}dx$$

$$\frac{900}{3\times 2EI}\left[x^3\right]_0^4 + \frac{900}{3\times 4EI}\left[x^3\right]_0^4 = \frac{14400}{EI}$$

$$\frac{1}{2} \times P \times \delta = \frac{14400}{EI}$$

$$\delta = \frac{\frac{14400}{EI} \times 2}{P = 60} = \frac{480}{EI}$$

Q-2 Determine the vertical deflection of point (c) in the frame. $E = 200\ kN/mm^2$ & $I = 30 \times 10^6 mm^4$



Sol

Portion	Origin	Limit	M_x
BC	C	0-3	$1 \times x = x$
AB	B	0-4	3

$$U = \int_0^3 \frac{x^2}{2EI} dx + \int_0^4 \frac{3^2}{2 \times 2EI} dx$$

$$U = \frac{13.5}{EI}$$

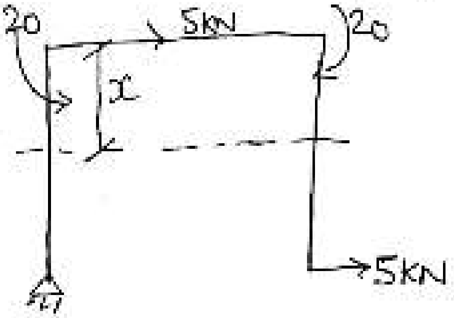
$$\frac{1}{2} \times P \times \delta = \frac{13.5}{EI}$$

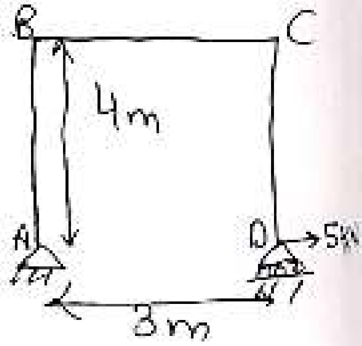
$$\delta = \frac{13.5}{EI} \times \frac{2}{1} = \frac{27}{200 \times 10^6 \times 30 \times 10^{-6}} = \frac{27}{6000}$$

$$= \cancel{0.0045} \; 0.0045m$$

$$= 4.5mm$$

Q.3 Determine the horizontal displacement of roller end D of portal frame $EI = 8000\ kN/m^2$





Portion	Origin	Limits	M_x
CD	D	0-4	$5x$
BC	C	0-3	20
AB	B	0-4	$20-5x$

$$\cancel{U = \int_0^4 \left(\frac{25x^3}{3}\right)_0^4 + \frac{1}{2EI}\left(400x\right)_0^3}$$

$$U = \int_0^4 \frac{(5x)^2}{2EI}dx + \int_0^3 \frac{(20)^2}{2EI}dx + \int_0^4 \frac{(20-5x)^2}{2EI}dx$$

$$U = \frac{1}{2EI}\left(\frac{25x^3}{3}\right)_0^4 + \frac{1}{2EI}\left(400x\right)_0^3 + \frac{1}{2EI}\left[400x + \frac{25x^3}{3} - \frac{200x^2}{2}\right]_0^4$$

$$= \frac{266.67}{EI} + \frac{600}{EI} + \frac{1}{2EI}\left(1600 + \frac{25 \times 64}{3} - 1600\right)$$

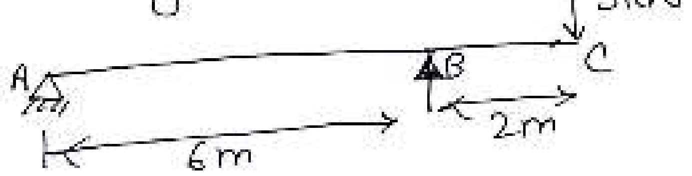
$$U = \frac{433.33}{EI}$$

$$\frac{1}{2} \times P \times \delta = \frac{433.33}{EI}$$

$$\delta = 0.0567 m$$

$$\delta = 56.7 mm$$

Q.3) Determine the vertical deflection at free end in overhanging beam. Assume constant EI. Use castigliano's method.



Sol : $P = 3 KN$

$$R_A + R_B = P$$

$$\Sigma M_A = 0, \quad R_B \times 6 - P \times 8 = 0$$

$$R_B = \frac{8P}{6} = \frac{4P}{3} (\uparrow) \qquad R_A = -\frac{P}{3} (\downarrow)$$

Portion	Origin	Limit	M_x
AB	A	0 – 6	$-\frac{P}{3}\times x$
BC	C	0 – 2	$-P\times x$

$$U = \int \frac{M^2}{2EI}dx = \int_0^6 \frac{P^2x^2}{9\times 2EI}dx + \int_0^2 \frac{P^2x^2}{2EI}dx$$

$$= \frac{P^2}{3\times 18EI}\left(x^3\right)_0^6 + \frac{P^2}{6EI}\left(x^3\right)_0^2$$

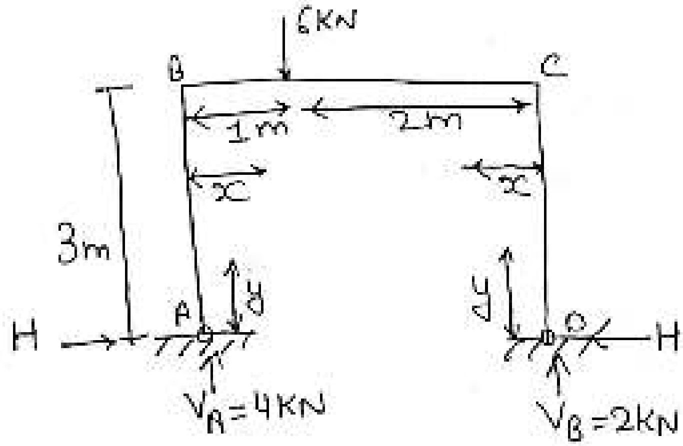
$$= \frac{P^2\times 6^3}{54EI} + \frac{P^2\times 2^3}{6EI} = \frac{5.33P^2}{EI}$$

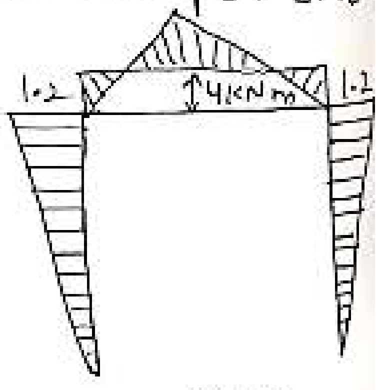
$$\delta = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P}\left(\frac{5.33P^2}{EI}\right) = \frac{2\times 5.33\times 3}{EI}$$

$$\delta = \frac{32}{EI}$$

Q.5 A portal frame ABCD is hinged at A & D and has rigid joints B & C. The frame is loaded. Using method of minimum strai energy, analyse the frame and plot BMD

Sol





BMD

$$\frac{\partial U}{\partial H} = 0 = \frac{\partial U_{AB}}{\partial H} + \frac{\partial U_{BC}}{\partial H} + \frac{\partial U_{DC}}{\partial H} \longrightarrow ①$$

Taking moments at D = $V = \frac{6 \times 2}{3} = 4\,kN$

$V_D = 6 - 4 = 2\,kN$

Member	M	$\partial M / \partial H$	Limit
AB	$-Hy$	$+y$	0-3
DC	$+Hy$	$+y$	0-3
BE	$-4x+3H$	$+3$	0-1
DE	$-2x+3H$	-3	0-2

For AB = $\frac{\partial U_{AB}}{\partial H} = \frac{1}{EI}\int_0^3 (Hy)(y)\,dy = \frac{H}{3EI}(3)^3 = \frac{9H}{EI}$

for DC = $\frac{\partial U_{DC}}{\partial H} = \frac{1}{EI}\int_0^3 (Hy)(y)\,dx = \frac{H}{3EI}(3)^3 = \frac{9H}{EI}$

for BC = $\frac{\partial U_{BC}}{\partial H} = \frac{1}{EI}\int_0^1 (-4x+3H)(+3)\,dx$

$$+ \int_0^2 (-2x+3H)(+3x)\,dx$$

$$= \frac{1}{EI}\left[(-6+9H) + (-12+18H)\right]$$

$$= \frac{9}{EI}(-2+3H)$$

Substituting values in eqn ①

$$\frac{9H}{EI} + \frac{9H}{EI} + \frac{9}{EI}(-2+3H) = 0$$

$$\boxed{H = 0.4\,kN}$$

Hence $M_A = 0$

$M_B = 3H = 0.4 \times 3 = 1.2\,kNm$

$M_C = 3H = 1.2\,kNm$

$M_D = 0$

- <u>Analysis of frame with redundant member</u>

$$\frac{\partial U}{\partial T_1} = 0 \quad , \quad \frac{\partial U}{\partial T_2} = 0 \quad etc$$

$$U = \frac{P^2 L}{2AE}$$

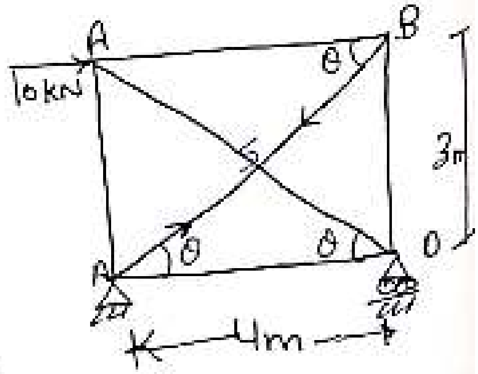
$$\frac{\partial U}{\partial T} = \sum_1^n P \frac{\partial P}{\partial T} \frac{L}{AE} = 0$$

$$\boxed{P = aT + bW}$$

T = redundant force

$$\frac{\partial P}{\partial T} = 0$$

Q.1 Find the force in the member BC of frame loaded as shown. All the members have same cross-sectional area.



Sol = ~~Static~~ Degree of reduntacy

$= m + r - 2J$

$= 6 + 3 - 2 \times 4 = 1$

$\sin\theta = \frac{3}{5} = 0.6 \quad , \quad \cos\theta = \frac{4}{5} = 0.8$

- At B resolving vertically

$P_{BD} = T\sin\theta = 0.6T$ (comp)

Resolving horizontally

$P_{BA} = T\cos\theta = 0.8T$ (comp)

- At A resolving ~~ver~~ horizontally

$P_{AB} = \frac{1}{\cos\theta}(10 - P_{AB})$

$= 1.25(10 - 0.8T) = 12.5 - T$ (comp)

Resolving horizontally

$P_{AC} = P_{AD}\sin\theta$

$= 0.6(12.5 - T) = 7.5 - 0.6T$ (Tension)

- Resolving horizontally at D

$P_{CD} = P_{AD}\cos\theta$

$= 10 - 0.8T$ (Tension)

Member	L(m)	P	$\frac{\partial P}{\partial T}$	$P\frac{\partial P}{\partial T}.L$
AB	4	$-0.8T$	-0.8	$2.56T$
BC	3	$-0.6T$	-0.6	$1.08T$
CD	4	$10-0.8T$	-0.8	$-32+2.56T$
AC	3	$7.5-0.6T$	-0.6	$-13.5+1.08T$
AD	5	$T-12.5T$	1	$-62.5+5T$
BC	5	T	1	$5T$
				$\boxed{-108+17.28T}$

$$\frac{\partial U}{\partial T} = 0 \quad \sum_1^b P\frac{\partial P}{\partial T}\frac{L}{AE} = 0$$

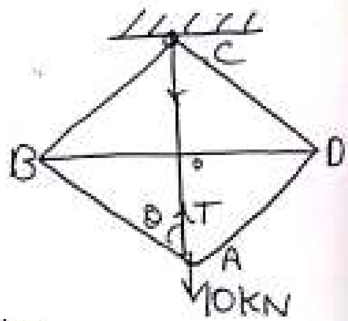
$$-108 + 17.28T = 0$$

$$\boxed{T = 6.25\text{ kN Tension}}$$

Scanned by CamScanner

Q.2 A frame work consists of six bars of uniform x-sectional area and hinged together to form a square with two diagnols, is suspended from one end as shown. At the opposite carrier a load of 10kN is suspended. Calculate the forces in all the member. The diagnol act independently.

Sol • Degree of redundancy



$m + r - 2J$

$6 + 3 - 2 \times 4 = 1$

• Treating member AC to be redundant replace it with tensile force T at joints A & C as shown.

$$\frac{\partial U}{\partial T} = \sum_1^n P \frac{\partial P}{\partial T} \frac{L}{AE} = 0$$

• Since A & E are same for all ~~section~~ members

$$\frac{\partial U}{\partial T} = \sum_1^n P \frac{\partial P}{\partial T} L = 0$$

• Calculation of stresses in the member

let length of side of square $= L$

diagnol $= L\sqrt{2}$

• Resolving $\Sigma H = 0$ at A

$P_{AB} = P_{AD}$

• Resolving vertically

$2P_{AB}\cos\theta + T = 10$

$$P_{AB} = P_{AD} = \frac{\sqrt{2}}{2}(10-T) = \frac{1}{\sqrt{2}}(10-T) \text{ tension}$$

• at C , Resolving Reaction at C = 10 kN

$$P_{CD} = P_{BC} = \frac{1}{\sqrt{2}}(10-T) \text{ tension}$$

• Resolve at B

$$P_{BD} = P_{BC}\cos\theta + P_{AB}\cos\theta$$

$$= \frac{2}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(10-T)\right] = (10-T) \text{ comp.}$$

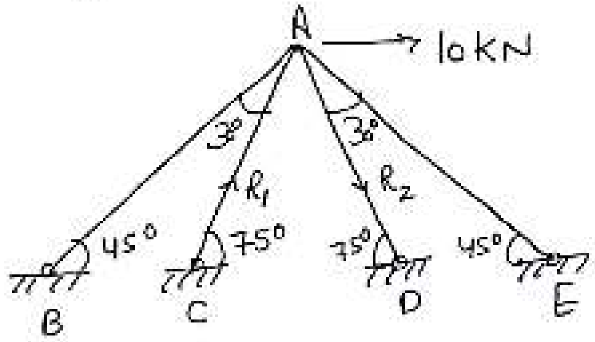
$$\rightarrow \sum_{1}^{n} P \frac{\partial P}{\partial T} L = (4.828T - 34.14)L = 0$$

$$T = 7.07 \text{ kN}$$

(+ve − tension) (−ve comp)

Member	L	P	$\frac{\partial P}{\partial T}$	$P\frac{\partial P}{\partial T} \cdot L$	Final stress (P)
AB	L	$\frac{1}{\sqrt{2}}(10-T)$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}(10-T)L$	+2.07
BC	L	"	"	"	+2.07
CD	L	"	"	"	+2.07
DA	L	"	"	"	+2.07
BD	$L\sqrt{2}$	$-10+T$	+1	$\sqrt{2}(T-10)L$	−2.93
AC	$L\sqrt{2}$	T	+1	$\sqrt{2}TL$	+7.07
				$\boxed{(4.828T-34.14)L}$	

Q.3 Find the forces in all members of frame. All bars are of same area of cross-section & are of same material.



Sol

- Degree of redundancy

$= m + r - 2J$

$= 4 + 8 - (2\times5) = 2$

- Thus frame is indeterminate to 2^{nd} degree
- Considering AB & AD are redundants, replace them with forces R_1 & R_2.
- It is assumed that AC carries a tensile force R_1 & AD = R_2 (comp).
- Consider the equilibrium of joint A

Resolving horizontally

$P_{AB}\cos45 + R_1\cos75 + R_2\cos75 + P_{AE}\cos45 = 0$

$0.707P_{AB} + 0.259(R_1+R_2) - 0.707\,P_{AE} = 10$

Resolving vertically

$P_{AB}\sin45 + R_1\sin75 + R_2\sin75° - P_{AE}\sin45° = 0$

$0.707\,P_{AB} + 0.966(R_1 - R_2) - 0.707\,P_{AE} = 0$ →

Solving ① & ②

$P_{AB} = 0.5R_2 - 0.866R_1 + 7.07$ (tension)

$P_{AE} = 0.5R_2 - 0.866R_1 + 7.07$ (comp)

Let ht of frame = L

Length of AB & AE = $L \operatorname{cosec} 45° = 1.414L$

" " AC & AD = $L \operatorname{cosec} 75° = 1.035L$

Member	L	P	$\frac{\partial P}{\partial R_1}$	$\frac{\partial P}{\partial R_2}$	$P\frac{\partial P}{\partial R_1}L$	$P\frac{\partial P}{\partial R_2}L$
AB	$1.414L$	$(0.5R_2 - 0.866R_1 + 7.07)$	-0.866	$+0.5$	$-(0.612R_2 - 1.06R_1 + 8.66)L$	$(0.354R_2 + 0.612R_1 + 5)L$
AC	$1.035L$	$+R_1$	$+1$	0	$1.035R_1L$	0
AD	$1.035L$	$-R_1$	0	-1	0	$1.035R_2L$
AE	$1.414L$	$0.866R_2 - 0.5R_1 - 7.07$	-0.5	0.866	$-(0.622R_2 - 0.354R_1 - 5)L$	$(1.06R_2 - 0.612R_1 - 8.66)L$

$$\frac{\partial U}{\partial R_1} = 0 = \sum_1^n P\frac{\partial P}{\partial R_1}L = (2.449R_1 - 1.224R_2 - 3.66)L \quad ①$$

$$\frac{\partial U}{\partial R_2} = 0 = \sum_1^n P\frac{\partial P}{\partial R_2}L = (2.449R_2 - 1.224R_1 - 3.66)L \quad ②$$

Solving ① & ② we get

$R_1 = 3\,KN$ (tensile)

$R_2 = 3\,KN$ (comp)

$P_{AB} = 6\,KN$ (tension)

$P_{AE} = 9\,KN$ (comp)